l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 respectively, such that

$$\left.\begin{aligned} l_1 \tau_{xx} + m_1 \tau_{yx} + n_1 \tau_{zx} &= l_1 \sigma_1 \\ l_1 \tau_{xy} + m_1 \tau_{yy} + n_1 \tau_{zy} &= m_1 \sigma_1 \\ l_1 \tau_{xz} + m_1 \tau_{yz} + n_1 \tau_{zz} &= n_1 \sigma_1 \end{aligned}\right\} \cdots\cdots (4)$$

$$\left.\begin{aligned} l_2 \tau_{xx} + m_2 \tau_{yx} + n_2 \tau_{zx} &= l_2 \sigma_2 \\ l_2 \tau_{xy} + m_2 \tau_{yy} + n_2 \tau_{zy} &= m_2 \sigma_2 \\ l_2 \tau_{xz} + m_2 \tau_{yz} + n_2 \tau_{zz} &= n_2 \sigma_2 \end{aligned}\right\} \cdots\cdots (5)$$

and another set of equations is obtained by replacing l_2, m_2, n_2 and σ_2 by l_3, m_3, n_3 and σ_3.

Now we shall prove that the three principal stress values $\sigma_1, \sigma_2, \sigma_3$ are real and that the corresponding principal stress directions are mutually orthogonal.

Multiply the three equations of (4) by l_2, m_2, n_2 respectively and adding,

$$l_1 l_2 \tau_{xx} + m_1 m_2 \tau_{yy} + n_1 n_2 \tau_{zz} + (l_2 m_1 + l_1 m_2)\tau_{xy} + (m_2 n_1 + m_1 n_2)\tau_{yz}$$
$$+ (n_2 l_1 + n_1 l_2)\tau_{zx} = \sigma_1 (l_1 l_2 + m_1 m_2 + n_1 n_2)$$ [considering the symmetry of $\tau_{xy} = \tau_{yx}$ etc.]

Similarly, multiplying the three equations of (5) by l_1, m_1, n_1 respectively and adding,

$$l_1 l_2 \tau_{xx} + m_1 m_2 \tau_{yy} + n_1 n_2 \tau_{zz} + (l_2 m_1 + l_1 m_2)\tau_{xy} + (m_2 n_1 + m_1 n_2)\tau_{yz}$$
$$+ (n_2 l_1 + n_1 l_2)\tau_{zx} = \sigma_2 (l_1 l_2 + m_1 m_2 + n_1 n_2)$$

Subtracting these two equations, we get

$$0 = (\sigma_1 - \sigma_2)(l_1 l_2 + m_1 m_2 + n_1 n_2) \cdots\cdots (6)$$

Similarly, we can obtain

$3O'_2$

$$0 = (\sigma_2 - \sigma_3)(l_2 l_3 + m_2 m_3 + n_2 n_3) \cdots (7)$$

and $$0 = (\sigma_1 - \sigma_3)(l_3 l_1 + m_3 m_1 + n_3 n_1) \cdots (8)$$

Let us now consider eq^n (3) which is a cubic eq^n in σ with roots σ_1, σ_2 and σ_3. Let us now assume that this equation has a complex root. Since this is a cubic equation ~~equation~~ with real co-efficient so another root must also be complex which will be the complex conjugate of the former. So the set of roots may be written as $\sigma_1 = \alpha + i\beta$, $\sigma_2 = \alpha - i\beta$, σ_3 where α, β and σ_3 are real numbers. So equation (5) can be written as

$$\left.\begin{aligned} l_2 \bar{\tau}_{xx} + m_2 \bar{\tau}_{yx} + n_2 \bar{\tau}_{zx} &= l_2 \bar{\sigma}_1 \\ l_2 \bar{\tau}_{xy} + m_2 \bar{\tau}_{yy} + n_2 \bar{\tau}_{zy} &= m_2 \bar{\sigma}_1 \\ l_2 \bar{\tau}_{xz} + m_2 \bar{\tau}_{yz} + n_2 \bar{\tau}_{zz} &= n_2 \bar{\sigma}_1 \end{aligned}\right\} \cdots (9)$$

Since $\sigma_2 = \bar{\sigma}_1$ and τ_{xx}, τ_{xy} etc. being real, $\tau_{xx} = \bar{\tau}_{xx}$, $\tau_{xy} = \bar{\tau}_{xy}$ etc. the coefficients of l_2, m_2, n_2 in eq^n(9) are complex conjugate of the co-efficients of l_1, m_1, n_1 in eq^n(4). So the values of l_2, m_2, n_2 determined from equation (9) are the complex conjugate of the values of l_1, m_1, n_1 determined from (4). So

if $\quad l_1 = a_1 + ib_1$, $m_1 = a_2 + ib_2$, $n_1 = a_3 + ib_3$ then

$$l_2 = a_1 - ib_1, \quad m_2 = a_2 - ib_2, \quad n_2 = a_3 - ib_3$$

$$\therefore l_1 l_2 + m_1 m_2 + n_1 n_2 = (a_1+ib_1)(a_1-ib_1) + (a_2+ib_2)(a_2-ib_2) + (a_3+ib_3)(a_3-ib_3)$$

$$= a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2 \neq 0.$$

$\therefore$ It follows that from eq^n (6) then $\quad \sigma_1 - \sigma_2 = 0$

i.e. $\quad \alpha + i\beta - \alpha + i\beta = 0$

or, $\quad 2i\beta = 0$

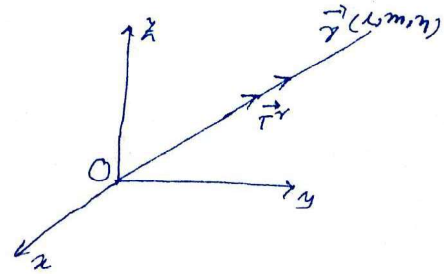
$\Rightarrow \beta = 0.$

This contradicts the original assumption that the roots are complex. So the assumption of the existence of complex root of eq^n(3) is not true, i.e. the roots $\sigma_1, \sigma_2, \sigma_3$ are all real.

If $\sigma_1 \neq \sigma_2 \neq \sigma_3$, then from eq^{n}s (6), (7), (8) we find that the principal stress directions are mutually orthogonal. If $\sigma_1 = \sigma_2 \neq \sigma_3$ then l_3, m_3, n_3 are fixed but we can not determine an infinite number of values of the d-cs (l_1, m_1, n_1) and (l_2, m_2, n_2) orthogonal to l_3, m_3, n_3.

If $\sigma_1 = \sigma_2 = \sigma_3$ then any set of orthogonal axes may be taken as principal axes.

Stress Invariants: 992



Let ox, oy, oz be a system of orthogonal axes w.r.t. which the stress tensor is

$$\begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}$$

Let $\vec{r}$ be the direction of principal stress at O. Then $\vec{T}^r$ is in the direction of $\vec{r}$. If l, m, n be the d-cs of this line and if σ be the magnitude of principal stress then

$\tau_{rx} = l\sigma, \quad \tau_{ry} = m\sigma, \quad \tau_{rz} = n\sigma$

i.e.

$$l\tau_{xx} + m\tau_{yx} + n\tau_{zx} = l\sigma \qquad \text{or,} \qquad l(\tau_{xx} - \sigma) + m\tau_{yx} + n\tau_{zx} = 0$$

$$l\tau_{xy} + m\tau_{yy} + n\tau_{zy} = m\sigma \qquad\qquad l\tau_{xy} + m(\tau_{yy} - \sigma) + n\tau_{zy} = 0$$

$$l\tau_{xz} + m\tau_{yz} + n\tau_{zz} = n\sigma \qquad\qquad l\tau_{xz} + m\tau_{yz} + n(\tau_{zz} - \sigma) = 0$$

eliminating l, m, n ; we have $\begin{vmatrix} \tau_{xx}-\sigma & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy}-\sigma & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz}-\sigma \end{vmatrix} = 0$

this is a cubic eq^n in σ.

This can be written as,
$$\sigma^3 - \Theta\sigma^2 + H\sigma - \Delta = 0 \quad \text{--- (A)}$$

where
$$\Theta = \tau_{xx} + \tau_{yy} + \tau_{zz} \quad \cdots (1)$$

$$H = \begin{vmatrix} \tau_{xx} & \tau_{yx} \\ \tau_{xy} & \tau_{yy} \end{vmatrix} + \begin{vmatrix} \tau_{xx} & \tau_{xz} \\ \tau_{zx} & \tau_{zz} \end{vmatrix} + \begin{vmatrix} \tau_{yy} & \tau_{yz} \\ \tau_{zy} & \tau_{zz} \end{vmatrix} \quad \text{-- (2)}$$

All the three roots of the eq^n (A) are real and they are the values of three principal stresses $\sigma_1, \sigma_2, \sigma_3$.
Since the principal stresses characterised the physical state of stress at a point they are independent of any co-ordinate of reference. Therefore

(a) $\tau_{xx} + \tau_{yy} + \tau_{zz}$ being equal to $\sigma_1 + \sigma_2 + \sigma_3$ is invariant under a co-ordinate transformation (1st Invariant).

(b) $\begin{vmatrix} \tau_{xx} & \tau_{yx} \\ \tau_{xy} & \tau_{yy} \end{vmatrix} + \begin{vmatrix} \tau_{xx} & \tau_{xz} \\ \tau_{zx} & \tau_{zz} \end{vmatrix} + \begin{vmatrix} \tau_{yy} & \tau_{yz} \\ \tau_{zy} & \tau_{zz} \end{vmatrix}$ being equal to

$\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$ is also invariant under co-ordinate transformation (2nd Invariant).

(c) $\begin{vmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{vmatrix}$ being equal to $\sigma_1\sigma_2\sigma_3$ is also invariant under co-ordinate transformation (3rd Invariant).

$\boxed{\text{SUM}}$ If the state of stress at any point of a body be given by $\tau_{xx} = \sigma_x = y^2 + \nu(x^2 - y^2)$

$$\tau_{yy} = \sigma_y = x^2 + \nu(y^2 - x^2)$$

$$\tau_{zz} = \sigma_z = x^2 + y^2$$

$\tau_{yz} = \tau_{zx} = 0$ and $\tau_{xy} = f(x, y)$ determine the expression for τ_{xy} in order that the stress distribution is in equilibrium in the absence of body force.

Sol^n: Eq^{n}s of equilibrium are (in absence of body force)

$$\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx} = 0 \quad \cdots (1)$$

$$\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy} = 0 \quad \cdots (2)$$

$$\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz} = 0 \quad \cdots (3)$$

From the given condition we have from (1) and (2),

$$2\gamma x + \frac{\partial}{\partial y} f(x,y) = 0 \quad \cdots (4)$$

$$2\gamma y + \frac{\partial}{\partial x} f(x,y) = 0 \quad \cdots (5)$$

Eq^n (3) is identically satisfied.

From (4) $\quad \frac{\partial}{\partial y} f(x,y) = -2\gamma x$

Int. w.r.t y, $\quad f(x,y) = -2\gamma xy + f_1(x) \quad \cdots (6)$

from (5), $\quad \frac{\partial}{\partial x} f(x,y) = -2\gamma y$

Int. w.r.t x, $\quad f(x,y) = -2\gamma xy + f_2(y) \quad \cdots (7)$

From (6) and (7) $\quad f_1(x) = f_2(y) = c$ where c is any constant

$$\tau_{xy} = f(x,y) = -2\gamma xy + c$$

Stress quadric of Cauchy: 293

Let Ox, Oy, Oz be a set of rectangular axes through O. Let us consider a quadric surface $\tau_{xx}x^2 + \tau_{yy}y^2 + \tau_{zz}z^2 + 2\tau_{yz}yz + 2\tau_{zx}zx + 2\tau_{xy}xy = \pm k^2 \quad \cdots (1)$
where k is constant and τ_{xx}, τ_{xy} etc. are the components of stress tensor at O reference to Ox, Oy and Oz as axes. This is called the stress quadric of Cauchy.

Let us make a transformation of axes according to the following scheme

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

The eq^n of quadric (1) referred to these new set of axes Ox', Oy', Oz' becomes

$$\tau_{xx}(l_1x'+l_2y'+l_3z')^2 + \tau_{yy}(m_1x'+m_2y'+m_3z')^2$$
$$+ \tau_{zz}(n_1x'+n_2y'+n_3z')^2 + 2\tau_{yz}(m_1x'+m_2y'+m_3z')(n_1x'+n_2y'+n_3z')$$
$$+ 2\tau_{zx}(n_1x'+n_2y'+n_3z')(l_1x'+l_2y'+l_3z') + 2\tau_{xy}(l_1x'+l_2y'+l_3z')$$
$$(m_1x'+m_2y'+m_3z') = \pm k^2 \quad \cdots (2)$$

The co-efficient of x'^2 in L.H.S. of the above eq^n is

$$\tau_{xx}l_1^2 + \tau_{yy}m_1^2 + \tau_{zz}n_1^2 + 2\tau_{yz}m_1n_1 + 2\tau_{zx}n_1l_1 + 2\tau_{xy}l_1m_1 = \tau_{x'x'}$$

similarly, the co-efficients of ~~y'^2 and z'^2~~ [By eq^n(7) of stress transformation law] ~~are respectively $\tau_{y'y'}$~~, y'^2 and z'^2 are $\tau_{y'y'}$ and $\tau_{z'z'}$ respectively.

The co-efficient of $2x'y'$ of (2) is

$$\tau_{xx}l_1l_2 + \tau_{yy}m_1m_2 + \tau_{zz}n_1n_2 + (m_2n_1 + m_1n_2)\tau_{yz} + (n_2l_1 + n_1l_2)\tau_{zx} + (l_2m_1 + l_1m_2)\tau_{xy}$$

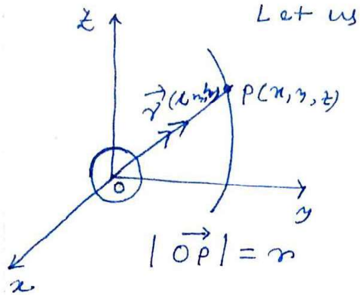
[By eq^n(8) of stress transformation law] $= \tau_{x'y'}$

similarly the co-efficient of $2y'z'$ and $2z'x'$ are respectively $\tau_{y'z'}$ and $\tau_{z'x'}$. So the eq^n of the quadric surface given by (1) when referred to Ox', Oy', Oz' as axes →

takes the form

$$\tau_{x'x'}x'^2 + \tau_{y'y'}y'^2 + \tau_{z'z'}z'^2 + 2\tau_{y'z'}y'z' + 2\tau_{z'x'}z'x' + 2\tau_{x'y'}x'y' = \pm K^2 \quad \ldots (3)$$

Now eq^n of the stress quadric referred to its principal axes as the co-ordinate axes OX, OY, OZ will obviously be of the form $\tau_{xx}x^2 + \tau_{yy}y^2 + \tau_{zz}z^2 = \pm K^2$ showing that the principal axes of the stress quadric are also the principal axes of stress at O.



Let us now study the other properties of the quadric surface given by eq^n ①. We draw any unit area through O. Let $\vec{r}$ be the unit normal to this elementary area where d-cs of $\vec{r}$ are l, m, n. Let us draw the radius vector $\vec{OP}$ to the quadric in this direction so that $P(x,y,z)$ is a point on the quadric and

$$|\vec{OP}| = r, \quad \therefore \frac{x}{r} = l, \quad \frac{y}{r} = m, \quad \frac{z}{r} = n.$$

The stress exerted by the material on the side towards which normal $\vec{r}$ is drawn on the opposite side across the unit area has for its component $\tau_{rx}, \tau_{ry}, \tau_{rz}$. The components of the stress in the direction of $\vec{r}$ i.e. the normal component of stress is given by,

$$l\tau_{rx} + m\tau_{ry} + n\tau_{rz} = l(l\tau_{xx} + m\tau_{yx} + n\tau_{zx}) + m(l\tau_{xy} + m\tau_{yy} + n\tau_{zy}) + n(l\tau_{xz} + m\tau_{yz} + n\tau_{zz})$$

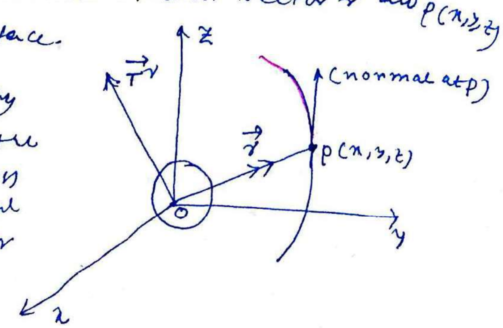
$$= l^2\tau_{xx} + m^2\tau_{yy} + n^2\tau_{zz} + 2lm\tau_{xy} + 2mn\tau_{yz} + 2nl\tau_{zx}$$

$$= \frac{1}{r^2}\left[x^2\tau_{xx} + y^2\tau_{yy} + z^2\tau_{zz} + 2xy\tau_{xy} + 2yz\tau_{yz} + 2zx\tau_{zx}\right]$$

$$= \pm\frac{K^2}{r^2} \quad [\text{by } ①]$$

So, if the quadric surface with centre at O given by eq^n ① can be drawn then the normal stress on any unit area through O can easily be derived by drawing the normal to the unit area. If the normal cuts the quadric surface at a distance r from O then $\pm\frac{K^2}{r^2}$ is the <u>normal stress</u>.

Another interesting property of the quadric surface given by eq^n ① is that the normal to this surface at the end of the radius vector $\vec{OP}$ is ~~in the direction~~ parallel to the stress vector $\vec{T}^r$ at O where $\vec{OP}$ is in the direction of unit vector $\vec{r}$ and $P(x,y,z)$ is a point on the quadric surface.



To prove this property we firstly note that the stress exerted by the material on the side of $\vec{r}$ across unit area at O of which normal to $\vec{r}$ to the material on the other side is $\vec{T}^r$. Its components are $\tau_{rx}, \tau_{ry}, \tau_{rz}$.

$\therefore$ D-rs of $\vec{T}^r$ are $\tau_{rx}, \tau_{ry}, \tau_{rz}$. Now eq^n of the quadric surface

(1) can be written as

$$f(x,y,z)=\tau_{xx}x^2+\tau_{yy}y^2+\tau_{zz}z^2+2\tau_{yz}yz+2\tau_{zx}zx+2\tau_{xy}xy \mp K^2=0$$

d.r.s of the normal to the surface at P

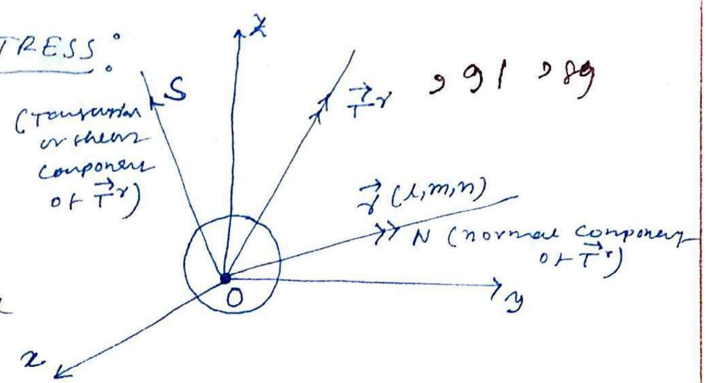
$$\frac{\partial f}{\partial x}=\tau_{xx}\cdot 2x+2\tau_{zx}z+2\tau_{yx}y = 2(x\tau_{xx}+y\tau_{yx}+z\tau_{zx})$$

$$\frac{\partial f}{\partial x}=2r(l\tau_{xx}+m\tau_{yx}+n\tau_{zx}) = 2r\tau_{rx}$$

$$\frac{\partial f}{\partial y}=2r\tau_{ry}, \quad \text{similarly, } \frac{\partial f}{\partial z}=2r\tau_{rz}$$

this shows that normal at P is parallel to the stress vector $\vec{T}^r$. So, if the stress quadric surface (1) be drawn at any point some properties of the stress at the centre of the quadric can at once be derived because of these facts the eqn (1) is called the eqn of stress quadric.

MAXIMUM SHEARING STRESS:



In order to determine the maximum shearing stress at any point O of the medium, we take the co-ordinate axes at O as Ox, Oy, Oz which are in the direction of principal stresses at O and let $\sigma_1, \sigma_2, \sigma_3$ be the corresponding stress values such that

$$\tau_{xx}=\sigma_1, \quad \tau_{yx}=0, \quad \tau_{zx}=0$$

$$\tau_{xy}=0, \quad \tau_{yy}=\sigma_2, \quad \tau_{zy}=0$$

$$\tau_{xz}=0, \quad \tau_{yz}=0, \quad \tau_{zz}=\sigma_3$$

we consider any arbitrary unit area through O with normal $\vec{r}$ determine by the d-cs (l,m,n). Then the stress exerted by the material on the side towards which normal $\vec{r}$ is drawn to the material on the ~~side towards~~ opposite side across the unit area has for its components $\tau_{rx}, \tau_{ry}, \tau_{rz}$ where

$$\left.\begin{aligned}\tau_{rx}&=l\tau_{xx}+m\tau_{yx}+n\tau_{zx}\\ \tau_{ry}&=l\tau_{xy}+m\tau_{yy}+n\tau_{zy}\\ \tau_{rz}&=l\tau_{xz}+m\tau_{yz}+n\tau_{zz}\end{aligned}\right\} \cdots\cdots (1)$$

The resultant stress can be written as

$$(\vec{T}^r)^2=\tau_{rx}^2+\tau_{ry}^2+\tau_{rz}^2 = l^2\sigma_1^2+m^2\sigma_2^2+n^2\sigma_3^2 \quad \cdots\cdots (2)$$

The normal stress N can be written as

$$N=l\tau_{rx}+m\tau_{ry}+n\tau_{rz} = l^2\sigma_1+m^2\sigma_2+n^2\sigma_3 \quad \cdots\cdots (3)$$

If S be the magnitude of the shearing stress, then

$$N^2+S^2=(\vec{T}^r)^2$$

or, $$S^2=(\vec{T}^r)^2-N^2=(l^2\sigma_1^2+m^2\sigma_2^2+n^2\sigma_3^2)-(l^2\sigma_1+m^2\sigma_2+n^2\sigma_3)^2 \cdots (4)$$

with rotation of the unit area about the point O the shearing stress S varies, being a function of two variables, say, l and m since we know, $n^2 = 1-l^2-m^2$

Using these relations, (4) becomes

$$S^2 = (\sigma_1^2-\sigma_3^2)l^2 + (\sigma_2^2-\sigma_3^2)m^2+\sigma_3^2 - \left[(\sigma_1-\sigma_3)l^2+(\sigma_2-\sigma_3)m^2+\sigma_3\right]^2 \quad \text{--- (5)}$$

To obtain maximum value of shearing stress we equate to zero the partial derivatives of S ~~which~~ with respect to l and m. At the values of l, m for which S is maximum, S^2 will also be maximum. Therefore we equate $\frac{\partial S^2}{\partial l}=0$, $\frac{\partial S^2}{\partial m}=0$ from which we obtain.

$$(\sigma_1^2-\sigma_3^2)l - 2\left[(\sigma_1-\sigma_3)l^2+(\sigma_2-\sigma_3)m^2+\sigma_3\right](\sigma_1-\sigma_3)l = 0 \quad \text{--- (6)}$$

$$(\sigma_2^2-\sigma_3^2)m - 2\left[(\sigma_1-\sigma_3)l^2+(\sigma_2-\sigma_3)m^2+\sigma_3\right](\sigma_2-\sigma_3)m = 0 \quad \text{--- (7)}$$

Consider the most general case when σ_1, σ_2 and σ_3 are all different. Then dividing eq^n (6) by $(\sigma_1-\sigma_3)$ and eq^n (7) by $(\sigma_2-\sigma_3)$ we obtain

$$\left\{(\sigma_1-\sigma_3) - 2\left[(\sigma_1-\sigma_3)l^2+(\sigma_2-\sigma_3)m^2\right]\right\}l = 0 \quad \text{--- (8)}$$

$$\left\{(\sigma_2-\sigma_3) - 2\left[(\sigma_1-\sigma_3)l^2+(\sigma_2-\sigma_3)m^2\right]\right\}m = 0 \quad \text{--- (9)}$$

We have two equations of the third degree in l and m. Accordingly we shall obtain three solutions of l, m and n. The first and the simplest solution is $l=0$, $m=0$, $n=1$ corresponding to this value of l, m and n. We find from equation (5) that $S=0$ this verifies the fact that plane element normal to the principal direction of stress is stress free. Thus the minimum value of $|S|$ is associated with the principal directions. Since we are seeking for maximum shearing stress, so discarding this value of l, m, n we have other three values (i) $l\neq 0$, $m=0$, (ii) $l=0$, $m\neq 0$ (iii) $l\neq 0$ $m\neq 0$. The third case is impossible since cancelling l and m out of equation (8) and (9) respectively and substract-ing the resulting equation we immediately obtained $\sigma_1=\sigma_2$ which contradicts our original assumption then $\sigma_1\neq\sigma_2\neq\sigma_3$

Next consider the 1st case, $l\neq 0$, $m=0$; then from eq^n (8) we obtained $(\sigma_1-\sigma_3)(1-2l^2)=0$ which gives

$$l=\pm\frac{1}{\sqrt{2}},\ m=0,\ n=\pm\frac{1}{\sqrt{2}} \quad \text{--- (10)}$$

When we consider the 2nd case, $l=0$, $m\neq 0$, then from eq^n (9) we obtained $(\sigma_2-\sigma_3)(1-2m^2)=0$ which gives

$$l=0,\ m=\pm\frac{1}{\sqrt{2}},\ n=\pm\frac{1}{\sqrt{2}} \quad \text{--- (11)}$$

If at the outset we had eliminated, say, m instead of n from eq^n (4) and repeated the analysis we would have obtained additionally one more solution, $l = \pm\frac{1}{\sqrt{2}}$, $m = \pm\frac{1}{\sqrt{2}}$, $n = 0$ --- (12)

The extreme values of shear stress can be obtained by substituting these values of d-cs in (5) by the help of (10) and (5), we obtain

$$(S_1^2)_{max.} = \frac{\sigma_1^2+\sigma_3^2}{2} - \frac{(\sigma_1+\sigma_3)^2}{4} = \frac{2(\sigma_1^2+\sigma_3^2)-(\sigma_1+\sigma_3)^2}{4} = \frac{(\sigma_1-\sigma_3)^2}{4}$$

i.e. $(S_1)_{extremum} = \pm\frac{\sigma_1-\sigma_3}{2}$

Similarly, by the help of (11) and (5)

$$(S_2)_{extremum} = \pm\frac{\sigma_2-\sigma_3}{2}$$

and by the help of (12) and (5),

$$(S_3)_{extremum} = \pm\frac{\sigma_1-\sigma_2}{2}$$

If $\sigma_1 > \sigma_2 > \sigma_3$, then the maximum value of $|S|$ is

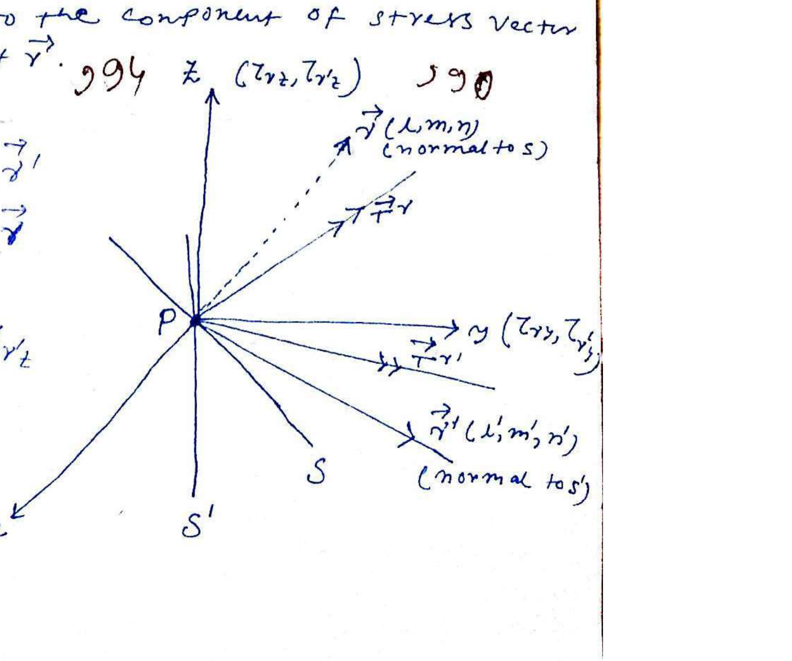
$$|S|_{max} = \frac{1}{2}(\sigma_1-\sigma_3)$$

In this case we find from eq^n(10) that the maximum shearing stress acts on the surface elements containing the y-principal axis and bisecting the angle between x and z axes.

Thus we conclude that the maximum shearing stress is equal to the one half the difference between the greatest and least normal stresses and acts on the plane that bisects the angle between the directions of the largest and smallest principal stresses.

THEOREM

Let surface elements S and S' with unit normals $\vec{\gamma}$ and $\vec{\gamma}'$ pass through the point P. Show that the component of stress vector $\vec{T}^{\gamma}$ (acting on S) in the direction of $\vec{\gamma}'$ is equal to the component of stress vector $\vec{T}^{\gamma'}$ (acting on S') in the direction of $\vec{\gamma}$.



proof: We have to prove

Component of $\vec{T}^{\gamma}$ in the direction of $\vec{\gamma}'$

= component of $\vec{T}^{\gamma'}$ in the direction of $\vec{\gamma}$

i.e. to prove

$$l'\tau_{\gamma x} + m'\tau_{\gamma y} + n'\tau_{\gamma z} = l\tau_{\gamma' x} + m\tau_{\gamma' y} + n\tau_{\gamma' z}$$

Now $l\tau_{\gamma' x} + m\tau_{\gamma' y} + n\tau_{\gamma' z}$

$$= l(l'\tau_{xx} + m'\tau_{yx} + n'\tau_{zx})$$
$$+ m(l'\tau_{xy} + m'\tau_{yy} + n'\tau_{zy})$$
$$+ n(l'\tau_{xz} + m'\tau_{yz} + n'\tau_{zz})$$

or, $l\tau_{r'x} + m\tau_{r'y} + n\tau_{r'z} = l'(l\tau_{xx} + m\tau_{yx} + n\tau_{zx})$

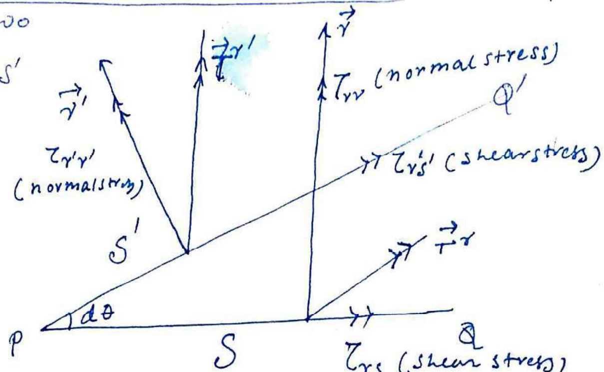
$$+ m'(l\tau_{xy} + m\tau_{yy} + n\tau_{zy})$$

$$+ n'(l\tau_{xz} + m\tau_{yz} + n\tau_{zz})$$

$\Rightarrow l\tau_{r'x} + m\tau_{r'y} + n\tau_{r'z} = l'\tau_{rx} + m'\tau_{ry} + n'\tau_{rz}$ [proved]

Hence show that the normal stress has a stationary value (max or min) when the shear stress is zero. 2 94

Let PQ and PQ' are two surface elements S and S' whose normal are in the direction of $\vec{r}$ and $\vec{r}'$ respectively. Angle between PQ and PQ' is a small angle $d\theta$. τ_{rr} and τ_{rs} are



the normal and shear components of stress on S and $\tau_{r'r'}$ and $\tau_{r's'}$ are the normal and shearing components of stress on S'. Obviously, $\tau_{r'r'}$ and $\tau_{r's'}$ are different from τ_{rr} and τ_{rs} and depend on θ. So, let us have the relation as

$\tau_{r'r'} = \tau_{rr} + \frac{\partial}{\partial\theta}\tau_{rr}\,d\theta$ and $\tau_{r's'} = \tau_{rs} + \frac{\partial}{\partial\theta}.\tau_{rs}\,d\theta$

we know that

Components of $\vec{T}^{r'}$ in the direction $\vec{r}$ = Component of $\vec{T}^{r}$ in the direction of $\vec{r}'$

$\therefore \tau_{r'r'}\cos d\theta + \tau_{r's'}\sin d\theta = \tau_{rr}\cos d\theta - \tau_{rs}\sin d\theta$

or, $\left(\tau_{rr} + \frac{\partial}{\partial\theta}\tau_{rr}\,d\theta\right)\cos d\theta + \left(\tau_{rs} + \frac{\partial}{\partial\theta}\tau_{rs}\,d\theta\right)\sin d\theta$

$$= \tau_{rr}\cos d\theta - \tau_{rs}\sin d\theta$$

$\therefore \frac{\partial}{\partial\theta}\tau_{rr}\cos d\theta + \frac{\partial\tau_{rs}}{\partial\theta}.\sin d\theta = 2\tau_{rs}\frac{\sin d\theta}{d\theta}$

Taking limit as $d\theta \to 0$, we get

$\frac{\partial}{\partial\theta}\tau_{rr} = -2\tau_{rs}$ i.e. rate of change of normal component of stress with θ equals to $-2 \times$ shearing component of stress.

If $\tau_{rs} = 0$, then $\frac{\partial}{\partial\theta}\tau_{rr} = 0$

i.e. τ_{rr} is stationary.

Ex. 1 w.r.t. the frame of reference oxyz, let the state of stress be
$$[\tau_{ij}] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
. Determine the principal stresses and their associated directions. Also check the invariance of I_1, I_2, I_3.

Solution For this state of stress 291

$$I_1 = \tau_{11} + \tau_{22} + \tau_{33} = 1+1+1 = 3$$

$$I_2 = \begin{vmatrix} \tau_{11} & \tau_{21} \\ \tau_{12} & \tau_{22} \end{vmatrix} + \begin{vmatrix} \tau_{22} & \tau_{32} \\ \tau_{23} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{31} \\ \tau_{13} & \tau_{33} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = -3$$

$$I_3 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 - 2(1) + 1(1) = -1$$

Let σ be the principal stress value, then the eqn for determining σ is
$$\begin{vmatrix} 1-\sigma & 2 & 1 \\ 2 & 1-\sigma & 1 \\ 1 & 1 & 1-\sigma \end{vmatrix} = 0$$

or, $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

or $\sigma^3 - 3\sigma^2 - 3\sigma + 1 = 0$

or $(\sigma^3+1) - 3\sigma(\sigma+1) = 0$

or $(\sigma+1)(\sigma^2 - 4\sigma + 1) = 0$

$\Rightarrow \sigma_1 = -1, \quad \sigma_2 = 2+\sqrt{3}, \quad \sigma_3 = 2-\sqrt{3}$

With a set of axes chosen along the principal axes the stress matrix will have the form

$$[\tau_{i'j'}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 2-\sqrt{3} \end{bmatrix}$$

Hence $I_1' = -1+2+\sqrt{3}+2-\sqrt{3}$
$= 3$

$$I_2' = \begin{vmatrix} -1 & 0 \\ 0 & 2+\sqrt{3} \end{vmatrix} + \begin{vmatrix} 2+\sqrt{3} & 0 \\ 0 & 2-\sqrt{3} \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 0 & 2-\sqrt{3} \end{vmatrix}$$

$$= -\cancel{2} - \cancel{\sqrt{3}} + \cancel{4} - 3 - \cancel{2} + \cancel{\sqrt{3}} = -3$$

$$I_3' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 2-\sqrt{3} \end{bmatrix} = (-1)(2+\sqrt{3})(2-\sqrt{3})$$
$$= -1$$

This checks the invariants of I_1, I_2 and I_3.

Direction of principal stress:

If l_1, m_1, n_1 be the d-cs of principal axis corresponding to the principal stress $\sigma_1 = -1$, then

$$l_1 + 2m_1 + n_1 = -1 \cdot l_1 \qquad\qquad l_1\tau_{xx} + m_1\tau_{yx} + n_1\tau_{zx} = l_1\sigma_1$$

$$2l_1 + m_1 + n_1 = -1 \cdot m_1 \qquad\qquad l_1\tau_{yx} + m_1\tau_{yy} + n_1\tau_{zy} = m_1\sigma_1$$

$$l_1 + m_1 + n_1 = -1 \cdot n_1 \qquad\qquad l_1\tau_{xz} + m_1\tau_{yz} + n_1\tau_{zz} = n_1\sigma_1$$

or $2l_1 + 2m_1 + n_1 = 0$

$2l_1 + 2m_1 + n_1 = 0$

$l_1 + m_1 + 2n_1 = 0$

solving the last two equations, we have

$$\frac{l_1}{3} = \frac{m_1}{-3} = \frac{n_1}{0} \quad \text{or} \quad \frac{l_1}{1} = \frac{m_1}{-1} = \frac{n_1}{0} = \frac{1}{\sqrt{2}}$$

so, $l_1 = \frac{1}{\sqrt{2}}$, $m_1 = -\frac{1}{\sqrt{2}}$, $n_1 = 0$

Again if l_2, m_2, n_2 be the d-cs of the principal axis corresponding to the principal stress $\sigma_2 = 2+\sqrt{3}$ then

$$l_2 + 2m_2 + n_2 = l_2(2+\sqrt{3})$$

$$2l_2 + m_2 + n_2 = m_2(2+\sqrt{3})$$

$$l_2 + m_2 + n_2 = n_2(2+\sqrt{3})$$

or $l_2(-1-\sqrt{3}) + 2m_2 + n_2 = 0$

$2l_2 + (-1-\sqrt{3})m_2 + n_2 = 0$

$l_2 + m_2 + (-1-\sqrt{3})n_2 = 0$

by cross-multiplication of the last two eq^s, we have

$$\frac{l_2}{3+\sqrt{3}} = \frac{m_2}{3+\sqrt{3}} = \frac{n_2}{2\sqrt{3}} \quad \text{or} \quad \frac{l_2}{\sqrt{3}+1} = \frac{m_2}{\sqrt{3}+1} = \frac{n_2}{2} = \frac{1}{\sqrt{12+4\sqrt{3}}}$$

$$= \frac{1}{2\sqrt{\sqrt{3}(\sqrt{3}+1)}}$$

$\therefore l_2 = \frac{1}{2}\left(1+\frac{1}{\sqrt{3}}\right)^{1/2}$

$m_2 = \frac{1}{2}\left(1+\frac{1}{\sqrt{3}}\right)^{1/2}$

$n_2 = \frac{1}{(3+\sqrt{3})^{1/2}}$

Now, if $\vec{N_1} = (l_1, m_1, n_1)$, $\vec{N_2} = (l_2, m_2, n_2)$ and $\vec{N_3} = (l_3, m_3, n_3)$

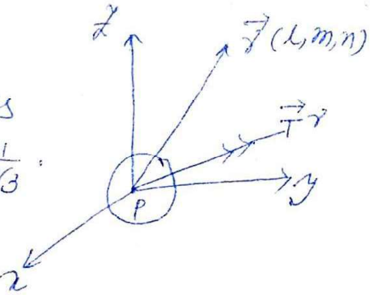
then $\vec{N_3} = \vec{N_1} \times \vec{N_2}$

$$\vec{N_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2}\left(1+\frac{1}{\sqrt{3}}\right)^{1/2} & \frac{1}{2}\left(1+\frac{1}{\sqrt{3}}\right)^{1/2} & \frac{1}{(3+\sqrt{3})^{1/2}} \end{vmatrix}$$

$$= \vec{i}\left[-\frac{1}{\sqrt{2}\,(3+\sqrt{3})^{1/2}}\right] + \vec{j}\left[-\frac{1}{\sqrt{2}\,(3+\sqrt{3})^{1/2}}\right] + \vec{k}\left[\frac{1}{\sqrt{2}}\left(1+\frac{1}{\sqrt{3}}\right)^{1/2}\right]$$

Ex: 2 At a point P in a body $\sigma_x = 10000\ kgf/cm^2$, $\sigma_y = -5000\ kgf/cm^2$, $\sigma_z = -5000\ kgf/cm^2$ and $\tau_{xy} = \tau_{yz} = \tau_{zx} = 10000\ kgf/cm^2$. determine the normal and shearing stresses on a plane which is equally inclined to all the three axes.
(d-cs equal)

Sol^n If $\vec{\nu}$ be the unit normal to the unit area at P, then d-cs of the normal $\vec{\nu}$ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.



993

Now $\tau_{\nu x} = l\tau_{xx} + m\tau_{yx} + n\tau_{zx}$

$= \frac{1}{\sqrt{3}}(\sigma_x + \tau_{yx} + \tau_{zx}) = \frac{1}{\sqrt{3}}(10000 + 10000 + 10000)$

$= 10000\sqrt{3}$

$\tau_{\nu y} = l\tau_{xy} + m\tau_{yy} + n\tau_{zy} = \frac{1}{\sqrt{3}}(\tau_{xy} + \sigma_y + \tau_{zy})$

$= \frac{1}{\sqrt{3}}(10000 - 5000 + 10000)$

$= 5000\sqrt{3}$

$\tau_{\nu z} = l\tau_{xz} + m\tau_{yz} + n\tau_{zz} = \frac{1}{\sqrt{3}}(\tau_{xz} + \tau_{yz} + \sigma_z)$

$\boxed{l^2+m^2+n^2=1,\quad l=m=n=\frac{1}{\sqrt{3}}}$

$= \frac{1}{\sqrt{3}}(10000 + 10000 - 5000)$

$= 5000\sqrt{3}$

If σ_ν be the normal component of stress vector on the unit area at P, of which the unit normal is $\vec{\nu}$, then

$\sigma_\nu = l\tau_{\nu x} + m\tau_{\nu y} + n\tau_{\nu z} = \frac{1}{\sqrt{3}}(10000\sqrt{3} + 5000\sqrt{3} + 5000\sqrt{3})$

$= 20000$

Also $|\vec{T}^{\nu}|^2 = \tau_{\nu x}^2 + \tau_{\nu y}^2 + \tau_{\nu z}^2 = 3\times10^8 + 3\times25\times10^6 + 3\times25\times10^6$

$= 450\times10^6$

If θ be the angle between the direction of $\vec{T}^{\nu}$ and $\vec{\nu}$, then

$\sigma_\nu = T^{\nu}\cos\theta$ and shearing or tangential stress $\tau_\nu = \vec{T}^{\nu}\sin\theta$

$\therefore \sigma_r^2 + \tau_r^2 = T_r^2$

$\Rightarrow \tau_r^2 = T_r^2 - \sigma_r^2$

$\Rightarrow \tau_r = \sqrt{T_r^2 - \sigma_r^2}$

$= \sqrt{450\times 10^6 - 400\times 10^6}$

$= 5000\sqrt{2}$

EX: 3 (Home Task)

The stress tensor at a point P is given by w.r.t. the axes $Oxyz$ by the values $\tau_{ij} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ determine the principle stress values and their direction w.r.t the axes $Ox'y'z'$.

Sol^n $\Rightarrow$ Let σ be the principal stress value, then the eq^n for determining σ is $\begin{vmatrix} 3-\sigma & 1 & 1 \\ 1 & -\sigma & 2 \\ 1 & 2 & -\sigma \end{vmatrix} = 0$

$\Rightarrow (3-\sigma)(\sigma^2 - 4) - (-\sigma - 2) + (2+\sigma) = 0$

$\Rightarrow (\sigma+2)\{(3-\sigma)(\sigma-2)+1+1\} = 0$

$\Rightarrow (\sigma+2)(3\sigma - \cancel{6} - \sigma^2 + 2\sigma + 2) = 0$

$\Rightarrow (\sigma+2)(5\sigma - 4 - \sigma^2) = 0$

$\Rightarrow (\sigma+2)(\sigma-4)(1-\sigma) = 0$

$\Rightarrow \sigma_1 = -2,\ \sigma_2 = 4,\ \sigma_3 = 1$

If l_1, m_1, n_1 be the d-cs of principal axis corresponding to the principal stress $\sigma_1 = -2$, then

$5l_1 + m_1 + n_1 = 0$

$l_1 + 2m_1 + 2n_1 = 0$

$l_1 + 2m_1 + 2n_1 = 0$

by cross-multiplication of the above 1st two equations

$\frac{l_1}{0} = \frac{m_1}{-9} = \frac{n_1}{9}$ $\Rightarrow$ $\frac{l_1}{0} = \frac{m_1}{-1} = \frac{n_1}{1} = \frac{1}{\sqrt{2}}$

$\Rightarrow l_1 = 0,\ m_1 = -\frac{1}{\sqrt{2}},\ n_1 = \frac{1}{\sqrt{2}}$

$\frac{l_1}{0} = \frac{m_1}{1} = \frac{n_1}{-1} = \frac{1}{\sqrt{2}}$ $\Rightarrow l_1 = 0,\ m_1 = \frac{1}{\sqrt{2}},\ n_1 = -\frac{1}{\sqrt{2}}$

Again if l_2, m_2, n_2 be the d-cs of the principal axes corresponding the principal stress $\sigma_2 = 4$, then

$$-l_2 + m_2 + n_2 = 0$$

$$l_2 - 4m_2 + 2n_2 = 0$$

$$l_2 + 2m_2 - 4n_3 = 0$$

by cross-multiplication of the above last two equations, we have

$$\frac{l_2}{6} = \frac{m_2}{3} = \frac{n_2}{3}$$

$$\text{or} \quad \frac{l_2}{2} = \frac{m_2}{1} = \frac{n_2}{1} = \frac{1}{\sqrt{6}}$$

$$\therefore \quad l_2 = \frac{2}{\sqrt{6}}, \quad m_2 = \frac{1}{\sqrt{6}}, \quad n_2 = \frac{1}{\sqrt{6}}$$

Now if $\vec{N_1} = (l_1, m_1, n_1)$, $\vec{N_2} = (l_2, m_2, n_2)$ and $\vec{N_3} = (l_3, m_3, n_3)$

$$\text{Then} \quad N_3 = \vec{N_1} \times \vec{N_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{vmatrix}$$

$$2l_3 + m_3 + n_3 = 0$$

$$l_3 - m_3 + 2n_3 = 0.$$

$$\frac{l_3}{3} = \frac{m_3}{-3} = \frac{n_3}{-3} \qquad = \left[-\frac{1}{\sqrt{12}} - \frac{1}{\sqrt{12}}\right]\vec{i} + \left[\frac{\sqrt{2}}{\sqrt{6}}\right]\vec{j} + \left[\frac{\sqrt{2}}{\sqrt{6}}\right]\vec{k}$$

$$\frac{l_3}{1} = \frac{m_3}{-1} = \frac{n_3}{-1} = \frac{1}{\sqrt{3}} \qquad = -\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}$$

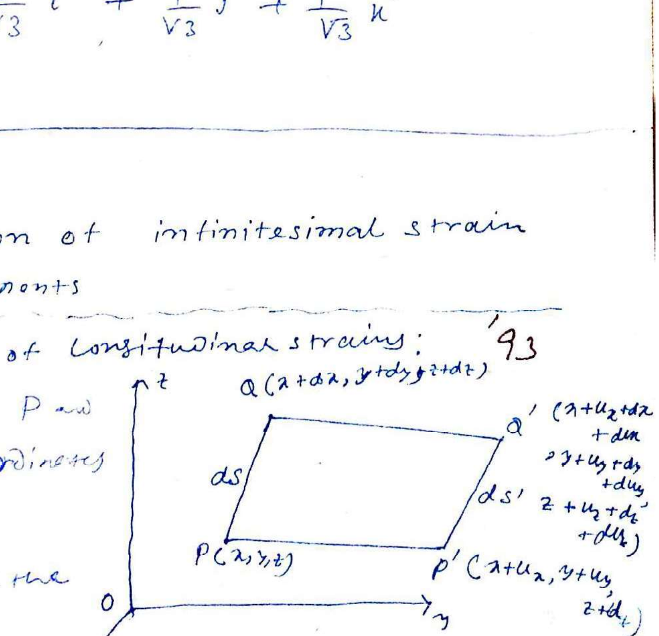
$$l_3 = \frac{1}{\sqrt{3}}, \quad m_3 = -\frac{1}{\sqrt{3}}, \quad n_3 = -\frac{1}{\sqrt{3}}$$

(+) Add After 7.2.96

Geometrical interpretation of infinitesimal strain components

(a) Geometrical interpretation of Longitudinal strains:

93



Let (x, y, z) be the co-ordinates of P and $(x+dx, y+dy, z+dz)$ be the coordinates of Q before deformation

Let $PQ = ds$, so it l, m, n be the d-cs of PQ, then

$$l = \frac{dx}{ds}, \quad m = \frac{dy}{ds}, \quad n = \frac{dz}{ds}$$

where $ds^2 = dx^2 + dy^2 + dz^2$

When deform slightly P occupies the position P' with co-ordinates $(x+u_x, y+u_y, z+u_z)$ and Q occupies the position Q' with co-ordinates